Computational Aspects of Singularity Theory

On occasion of the opening of the Graduiertenkolleg ’Experimental and Algorithmic Algebra’

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Aachen, 1. October 2010
Some Curves with and without singularities

\[ y^2 - x = 0 \]
\[ y^2 - x^2 - x^3 = 0 \]
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Some Surfaces with isolated and non-isolated singularities

\[ V(x^2 + y^2 - z^5) \]

\[ V(x^2 + y^2z - z^3) \]
Some Surfaces with isolated and non-isolated singularities

\[ V(x^2 + y^2 - z^5) \]
\[ V(x^2 + y^2 z - z^3) \]
\[ V(x^2 + y^4 - z^4 - 3x^2y^2) \]
\[ V(x^2 - y^4 + x^2z^4) \]
Where is $X = V(\langle f_1, \ldots, f_k \rangle) \subset \mathbb{A}^n_C$ singular?

Recall:

$X$ singular at $a \in \mathbb{A}^n_C \iff$ dimension of tangent space at $a$

$\iff$ exceeds dimension of $X$ at $a$

$\iff$ rank of jacobian

matrix of $(f_1, \ldots, f_k)$

$< n - \text{dimension of } X \text{ at } a$
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For our hypersurfaces:

$X$ singular at $a \iff \text{grad}(f)(a) = 0$
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For our hypersurfaces:

$X$ singular at $a \iff \text{grad}(f)(a) = 0$

In the pictures of the first slide:

- parabola non-singular, because $\frac{\partial f}{\partial x} = -1$
- cusp singular at $(0, 0)$, because

$$\frac{\partial f}{\partial x} = -3x^2, \quad \frac{\partial f}{\partial y} = 2y$$
Is $y^2 - x^3 = 0$ 'less singular' than $y^2 - x^5 = 0$?

$A_2 : y^2 - x^3 = 0$

$A_4 : y^2 - x^5 = 0$
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First idea of comparison: Consider ideals
\[ \langle f \rangle, \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle, \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \subset \mathbb{C}\{x, y\} \]
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or the quotients

$\mathbb{C}\{x, y\}/\langle f \rangle$ und $\mathbb{C}\{x, y\}/\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$
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Advantage: Use of tools of commutative algebra!
Some classical invariants of isolated singularities

Milnor-number of hypersurface $V(f)$:

$$\mu := \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$$

tjeurina-Zahl:

$$\tau := \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$$

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Computing Milnor and Tjurina number

Recall:
Computing $\dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/I$ for a 0-dimensional ideal $I$:
- compute a Groebner basis $G = \{g_1, \ldots, g_s\}$ of $I$ w.r.t. monomial ordering (in particular $L(G) = L(I)$)
- $\dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/L(G) = \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/I$
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In local situation:

- same computation, but w.r.t. local monomial ordering $(1 > x_i)$
- problem: not a well-ordering, hence no termination of Buchberger normal form
- solution: Mora normal form
Beginning of classification

For the plane curves of the first slide:

Milnor number: \( \mu(f) := \dim \mathbb{C} \{ x, y \}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \)

<table>
<thead>
<tr>
<th>curve</th>
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<th>A₂</th>
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**Multiplicity:** order of $f$,

i.e. smallest degree of a monomial in $f$

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many other invariants for plane curve or hypersurface singularities:

e.g. \( \delta \), monodromy, spectrum, . . .
Fibres of a Deformation of an $A_4$ Singularity

$y^2 - x^5 - t \cdot x^3$:

in blue: Fibre for $t = -1$ ($A_2$), $t = 0$ ($A_4$) and $t = 1$ ($A_2$)
Fibres of a Deformation of an $A_4$ Singularity

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again the fibres:
Deformations: Algorithmic Tasks

- What types of singularities appear in a given family? (classification)
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- What singularities can appear in a family with given special fibre? (adjacencies)
- What singularities of the same topological type can appear for a given special fibre?
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- Is it possible to construct moduli spaces for singularities with certain fixed invariants?
Some non-isolated Singularities

Singular locus can be of dimension 1 for surfaces!
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New Phenomena in higher dimension

Problems:
- singular locus does not need to be 0-dimensional
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example: desingularization
Blowing up, simplest case

Idea: replace a point in $\mathbb{A}_\mathbb{C}^2$ by a projective line
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Effect: more room for curves to become smooth
Blowing up, simplest case

Idea: replace a point in \( \mathbb{A}^2_{\mathbb{C}} \) by a projective line

Effect: more room for curves to become smooth
in pictures (only one chart):

\[
\begin{array}{c}
\text{Before} \\
\text{After}
\end{array}
\]
Computation of a blowing up

$X$ affine variety (ideal $I_X \subset K[x_1, \ldots, x_n]$)
$C$ smooth subvariety of $X$
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total transform \( X'_{total} \subset X \times \mathbb{P}^{k-1} \):
Computational Singularities

A. Frühbis-Krüger

Isolated Singularities
Introduction by pictures
Singular Points
Invarianten

Families of Singularities
Families
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Examples and Problems

Main Tool for Desingularization: Blowing up

Resolution of Singularities

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can be computed as preimage of $I_X$ under

$\Phi : K[x_1, \ldots, x_n, y_1, \ldots, y_k] \rightarrow K[x_1, \ldots, x_n, t]$

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algorithmically: Gröbner basis computation in \( n + k + 1 \)
variables w.r.t. an elimination ordering for \( t \)
One Blowing up does not suffice

blowing up $A_4$ at the origin (just one chart):
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blowing up $A_4$ at the origin (just one chart):

for Whitney’s umbrella a blowing up at the origin does not change the singularity (in one chart):
Idea of Resolution of Singularities

Theorem (Hironaka, 1964):
For every algebraic variety over a field of characteristic zero, a desingularization can be achieved by a finite sequence of suitable blow ups.
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Applications:
- properties/invariants of singularities
- birational invariants of varieties
- algebraic statistics (model selection in presence of hidden variables)
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Key task: choice of suitable centers
Tools: standard bases, ideal quotients, saturation, more specialized algorithms
Simplest Formulation of Desingularization

Given: algebraic variety $X$ (over $\mathbb{C}$)
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- proper birational morphism
  $$\pi : \tilde{X} \longrightarrow X$$
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such that $\text{Reg}(X) \cong \pi^{-1}(\text{Reg}(X))$
Desingularization as a Task of Computer algebra

1964  Hironaka’s non-constructive proof
ca. 1990  first algorithmic proof (Villamayor, Bierstone-Milman)
ca. 2001  first prototype-implementation (Bodnar-Schicho in Maple)
ca. 2004  first practically usable implementation (FK-Pfister in Singular)
Strategy for Choice of Center

always improve 'worst' locus
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simplest situation: plane curve singularities
singular locus is a finite set of points
always improve ’worst’ locus

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surface singularities:
  components of singular locus
    may be (singular) curves

When do we need to blow up in points, when in curves?
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    When do we need to blow up in points,
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How do we finde the ’worst’ locus?
'worst' locus is locus of maximal value of the controlling invariant
The controlling invariant

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Requirements for the controlling invariant:

(a) locus of maximal value is Zariski-closed (Zariski upper semicontinuity of the invariant)

(b) locus of maximal value is non-singular and has normal crossing with exceptional divisors
The controlling invariant

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Requirements for the controlling invariant:

(a) locus of maximal value is Zariski-closed (Zariski upper semicontinuity of the invariant)
(b) locus of maximal value is non-singular and has normal crossing with exceptional divisors
(c) maximal value can not increase under a blowing up
(d) decrease of maximal value is a measure for the improvement of the desingularization
Structure of the controlling invariant

$$(inv_d; inv_{d-1}; \ldots; inv_2)$$
Structure of the controlling invariant

\[(inv_d; inv_{d-1}; \ldots; inv_2)\]

central construction in Hironaka’s argument:
descent of ambient dimension
Structure of the controlling invariant

\((inv_d; inv_{d-1}; \ldots; inv_2)\)

central construction in Hironaka’s argument:
descent of ambient dimension

Basic idea using \(ord_\partial(f)\) as \(inv_d\):
Consider

\[ f = z^k + a_1(x)z^{k-1} + \cdots + a_k(x) \subset \mathbb{C}\{z, x\} \]
Structure of the controlling invariant

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central construction in Hironaka’s argument:
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Basic idea using \(ord_0(f)\) as \(inv_d\):
Consider

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We know: \(ord_0(f) = k \iff ord_0(a_i) \geq i \ \forall 1 \leq i \leq k\)
Structure of the controlling invariant

\[(\text{inv}_d; \text{inv}_{d-1}; \ldots; \text{inv}_2)\]

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Basic idea using \(\text{ord}_0(f)\) as \(\text{inv}_d\):
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We know: \(\text{ord}_0(f) = k \iff \text{ord}_0(a_i) \geq i \quad \forall 1 \leq i \leq k\)

hence: \(\text{ord}_0(f) = k \iff \text{ord}_0(\langle a_1^{k!}, a_2^2, \ldots a_k^{(k-1)!} \rangle) \geq k!\)
Main Problems of Desingularization

in characteristic zero:

- for singular loci of higher dimension
  point blow ups do not suffice
- singular locus has structure (e.g. singularities)
- unsuitable choice of centers can deteriorate the situation
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• Glueing of different charts

in positive characteristic:
• hypersurfaces of maximal contact need not exist
• order of an ideal may increase under blowing up